Worksheet # 2 Answers

1. a) $\sin\left(\frac{1}{3}\right) \approx \left(\frac{1}{3}\right) - \frac{\left(\frac{1}{3}\right)^3}{3!} = \frac{53}{162} \approx 0.327$ (this is S_2, the sum of the series considering only the first two terms)

b) The Maclaurin series for $\sin\left(\frac{1}{3}\right)$ is a convergent alternating series, so the error made in the approximation is less than the absolute value of the first term no considered:

$$R_2 = \left|\sin\left(\frac{1}{3}\right) - \frac{53}{162}\right| \le \frac{\left(\frac{1}{3}\right)^5}{5!} = \frac{1}{29{,}160}$$

This problem can also be done using Taylor's Theorem. To do so, we would have:

$R_3\left(\frac{1}{3}\right) = \frac{f^{(4)}(z)}{4!} \cdot \left(\frac{1}{3} - 0\right)^4$, where $R_3\left(\frac{1}{3}\right)$ represents the remainder when a polynomial of degree 3 is used in the approximation.

Since $f^{(4)}(x) = \sin x$ and $|\sin x| \le 1 \Rightarrow \left|f^{(4)}(z)\right| \le 1$

So: $\left|R_3\left(\frac{1}{3}\right)\right| \le \frac{1}{4!} \cdot \left(\frac{1}{3} - 0\right)^4 = \frac{1}{1{,}944}$

c) $\frac{9{,}539}{29{,}160} \le \sin\left(\frac{1}{3}\right) \le \frac{9{,}541}{29{,}160}$ or $0.32712 \le \sin\left(\frac{1}{3}\right) \le 0.32719$

If Taylor's Theorem was used for part (b), we would have:

$\frac{635}{1{,}944} \le \sin\left(\frac{1}{3}\right) \le \frac{637}{1{,}944}$ or $0.3266 \le \sin\left(\frac{1}{3}\right) \le 0.3276$

2. a) $e^{\frac{1}{4}} \approx 1 + \left(\frac{1}{4}\right) + \frac{\left(\frac{1}{4}\right)^2}{2!} + \frac{\left(\frac{1}{4}\right)^3}{3!} = \frac{493}{384} \approx 1.2838$

b) Using Taylor's Theorem: $R_3\left(\frac{1}{4}\right) = \frac{f^{(4)}(z)}{4!} \cdot \left(\frac{1}{4} - 0\right)^4$, where $R_3\left(\frac{1}{4}\right)$ represents the remainder when a polynomial of degree 3 is used in the approximation.

Since $f^{(4)}(x) = e^x$ which increases, and $0 < z < \frac{1}{4} \Rightarrow f^{(4)}(z) < e^{\frac{1}{4}} < 2$

So: $\left|R_3\left(\frac{1}{4}\right)\right| \le \frac{2}{4!} \cdot \left(\frac{1}{4} - 0\right)^4 = \frac{1}{3{,}072}$

The series for $e^{\frac{1}{4}}$ from part (a) is NOT alternating, so Taylor's theorem is the only method we can use to estimate the error made in the approximation.

c) $\frac{3{,}943}{3{,}072} \le e^{\frac{1}{4}} \le \frac{1{,}315}{1{,}024}$ or $1.2835 \le e^{\frac{1}{4}} \le 1.2841$

3. Let f be a function that has derivatives of all orders for all real numbers and for which $f(3) = -1$ and $f'(3) = 0$. For $n > 1$, the nth derivative of f at $x = 3$ is given by $f^{(n)}(3) = (-1)^n \cdot \frac{n!}{2^n}$.

a) Since $f'(3) = 0$ and $f''(3) = (-1)^2 \cdot \frac{2!}{2^2} = \frac{1}{2} > 0$ f has a local minimum at $x = 3$. The function has a horizontal tangent line and it is concave up (second derivative test.)

b) $f''(3) = \frac{1}{2}$ and $f^{(3)}(3) = -\frac{3}{4} \Rightarrow P_3(x) = -1 + \frac{1}{4}\cdot(x-3)^2 - \frac{1}{8}\cdot(x-3)^3$

c) $\frac{f^{(n)}(3)}{n!}\cdot(x-3)^n = \frac{(-1)^n \cdot \frac{n!}{2^n}}{n!}\cdot(x-3)^n = (-1)^n \cdot \frac{1}{2^n}\cdot(x-3)^n$

d) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n \cdot \frac{1}{2^{n+1}}\cdot(x-3)^{n+1}}{(-1)^n \cdot \frac{1}{2^n}\cdot(x-3)^n}\right| = |x-3|\lim_{n\to\infty}\left|\frac{1}{2}\right| = |x-3|\cdot\frac{1}{2} < 1 \Rightarrow |x-3| < 2$: Radius of convergence is 2.

e) $P_6(4) = -1 + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64}$. We can see that the resulting series is an alternating series. We can estimate the error using the remainder of the alternating series.
$|R_6| < a_7 = \frac{1}{2^7} = \frac{1}{128} \approx 0.008 < 0.01$

4. Let f be a function with derivatives of all orders for all real numbers. The third-degree Taylor polynomial for the function f about 1 is given by $T(x) = 6 + 2\cdot(x-1) - 4\cdot(x-1)^2 + 3\cdot(x-1)^3$.

a) $f'(1) = 2$ and $f''(1) = -4\cdot 2! = -8$.

b) Neither. Since $f'(1) = 2 > 0$ f is increasing at $x = 1$.

c) $f(0) \approx T(0) = 6 + 2\cdot(-1) - 4\cdot(-1)^2 + 3\cdot(-1)^3 = -3$

d) We can calculate the error committed in our approximation in part c):
$|R_3(0)| = \left|\frac{f^{(4)}(z)}{4!}\cdot(0-1)^4\right|$. Since $\left|f^{(4)}(x)\right| < 8$ for all values of x such that $0 \le x \le 1$:
$|R_3(0)| < \frac{8}{4!} = \frac{1}{3}$. Since the error is smaller than $\frac{1}{3} \Rightarrow -3 - \frac{1}{3} < f(0) < -3 + \frac{1}{3}$. And so $f(0) < -2$.

Worksheet # 3 Answers

(a) $f(2) = T(2) = 7$ $\dfrac{f''(2)}{2!} = -9$ so $f''(2) = -18$	$2 : \begin{cases} 1 : f(2) = 7 \\ 1 : f''(2) = -18 \end{cases}$
(b) Yes, since $f'(2) = T'(2) = 0$, f does have a critical point at $x = 2$. Since $f''(2) = -18 < 0$, $f(2)$ is a relative maximum value.	$2 : \begin{cases} 1 : \text{states } f'(2) = 0 \\ 1 : \text{declares } f(2) \text{ as a relative} \\ \quad \text{maximum because } f''(2) < 0 \end{cases}$
(c) $f(0) \approx T(0) = -5$ It is not possible to determine if f has a critical point at $x = 0$ because $T(x)$ gives exact information only at $x = 2$.	$3 : \begin{cases} 1 : f(0) \approx T(0) = -5 \\ 1 : \text{declares that it is not} \\ \quad \text{possible to determine} \\ 1 : \text{reason} \end{cases}$
(d) Lagrange error bound $= \dfrac{6}{4!}\lvert 0 - 2\rvert^4 = 4$ $f(0) \le T(0) + 4 = -1$ Therefore, $f(0)$ is negative.	$2 : \begin{cases} 1 : \text{value of Lagrange error} \\ \quad \text{bound} \\ 1 : \text{explanation} \end{cases}$

Let f be a function with derivatives of all orders for all real numbers. The third-degree Taylor polynomial for the function f about 1 is given by $T(x) = 6 + 2\cdot(x-1) - 4\cdot(x-1)^2 + 3\cdot(x-1)^3$.

a) $f'(1) = 2$ and $f''(1) = -4\cdot 2! = -8$.

b) Neither. Since $f'(1) = 2 > 0$ f is increasing at $x = 1$.

c) $f(0) \approx T(0) = 6 + 2\cdot(-1) - 4\cdot(-1)^2 + 3\cdot(-1)^3 = -3$

d) We can calculate the error committed in our approximation in part c):

$|R_3(0)| = \left|\dfrac{f^{(4)}(z)}{4!}\cdot(0-1)^4\right|$. Since $\left|f^{(4)}(x)\right| < 8$ for all values of x such that $0 \le x \le 1$:

$|R_3(0)| < \dfrac{8}{4!} = \dfrac{1}{3}$. Since the error is smaller than $\dfrac{1}{3} \Rightarrow -3 - \dfrac{1}{3} < f(0) < -3 + \dfrac{1}{3}$. And so $f(0) < -2$.

e) $G(x) = 6\cdot(x-1) + (x-1)^2 - \dfrac{4}{3}\cdot(x-1)^3 + C$

Since $G(1) = 5 = C \Rightarrow G(x) = 6\cdot(x-1) + (x-1)^2 - \dfrac{4}{3}\cdot(x-1)^3 + 5$

a) $e^{x^2} = 1 + \left(x^2\right) + \frac{\left(x^2\right)^2}{2!} + \frac{\left(x^2\right)^3}{3!} + \dots + \frac{\left(x^2\right)^n}{n!} + \dots$

or $e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots + \frac{x^{2n}}{n!} + \dots$

b) $\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots + (-1)^n \frac{(2x)^{2n}}{(2n)!} + \dots$

or $\cos 2x = 1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - \frac{64x^6}{6!} + \dots + (-1)^n \frac{(2x)^{2n}}{(2n)!} + \dots$

c) $f(x) = e^{x^2} + \cos 2x = 2 - x^2 + \frac{7}{6}x^4 + \frac{7}{90}x^6 + \dots$

d) $a_4 = \frac{f^{(4)}(0)}{4!} = \frac{7}{6}$, so $f^{(4)}(0) = 28$

e) Using Lagrange's form of the remainder: $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| = \left|R_4\left(\frac{1}{4}\right)\right| = \left|\frac{f^{(5)}(z)}{5!}\left(\frac{1}{4} - 0\right)^5\right|$

where $0 < z < \frac{1}{4}$.

The graph above clearly shows that $f^{(5)}(x) < 30$ on the interval $\left[0, \frac{1}{4}\right]$. Therefore we can bound the error as $\left|\frac{f^{(5)}(z)}{5!}\left(\frac{1}{4} - 0\right)^5\right| < \frac{30}{5!} \cdot \frac{1}{4^5} = \frac{1}{4{,}096}$, which shows that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{4000}$

Answers are not in order → A D E E E ☺